

# Predicting Wind Dispersal of Descending HPR Using a Monte-Carlo Method

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## Background

Last year saw a new altitude record in the UK with a flight exceeding 20,000 ft. As altitudes increase the distance that rockets will drift during descent will also increase, leading to the possibility of drift outside the range. A descending HPR, even under parachute, has sufficient kinetic energy to cause damage to property or serious injury<sup>1</sup>. It is thus highly desirable to keep rockets within the range in order to manage the risk to people and property not associated with rocket flying.

A model for predicting the dispersal pattern of descending rockets would be a useful tool for managing this risk. It would allow both fliers and RSOs to make an informed decision before a high altitude flight, and allow them to judge whether the flight could be made safely. The only known model was “*Splash*” by Apogee, and this is only available to US nationals. *Splash* is a full 6 degrees of freedom (6DOF) model of the complete flight from launch to landing. It incorporates failure modes and their probability in its estimate of likely landing sites, and is a comprehensive package for predicting the probability of landing outside the range.

In 2003 the author decided to start researching how such a model could be produced in the UK. A period of study and consultation on the technical issues of rocket drift with colleagues in aerospace companies and academia helped the author to discriminate between useful and impractical approaches. The current model, together with an approach for matching it to launch criteria, was proposed on the author’s website in 2004.

This paper reports on the first part of the author’s research. It proposes a model for predicting the dispersal of HPR descending from high altitudes. The model is based on readily available data and uses Monte-Carlo simulations of multiple descents to establish the statistics of the dispersal of landing sites.

The model has been implemented as an Excel spreadsheet “front end” with the model coded as a VBasic macro. This allows it to be used on any personal computer and, if required, easily modified. It does not, however, lend itself to easy implementation of more complex maths functions required to model the entire flight profile in 6DOF; the author’s preferred language for calculations in 6DOF would be MATLAB. The paper explains the principles of the model in sufficient detail to allow others to code it in the language of their choice.

It must be emphasised that the proposed model is experimental and, as yet, untested.

The paper starts by describing the basic principles of the model before expressing the equations of motion for the descending rocket. It continues by describing the

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<sup>1</sup> One source indicates that it takes about 20 Joules to fracture a skull; this corresponds to a 5kg rocket descending at 3 m/s (about 6 mph)

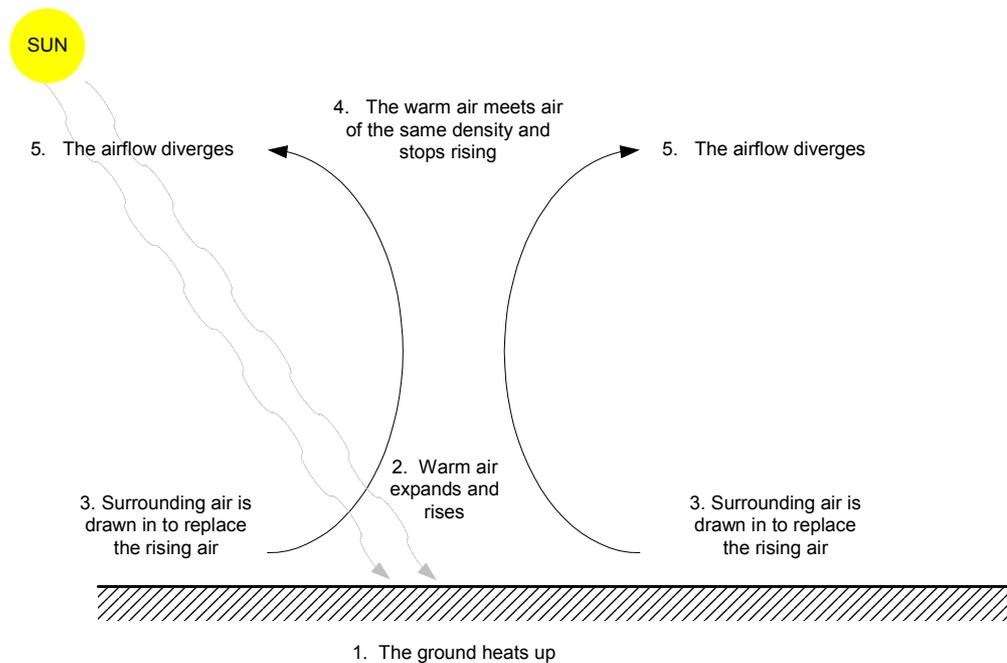
software and how to use it, before concluding with a short section on the future development of the model.

## The Wind

Wind is caused by differences in the local temperature of air. At a macro level, this causes high and low pressure regions in the atmosphere. Air flows from the high pressure systems to low pressure systems; however the flow between pressure systems is not constant in either direction or speed.<sup>2</sup>

The origins of winds can be illustrated by considering a low pressure system. In this system the Sun heats the ground which, in turn, warms the air above it. The warm air is less dense so it expands and rises. As the warm air rises it creates a pressure gradient which causes low level air to be drawn in from the higher pressure surrounding air.

The warm air continues to rise slowly, until it encounters air of the same density at an altitude of between 15,000 and 20,000 ft. At this altitude, it starts to diverge until it eventually sinks and completes the convection cycle. These effects can be seen in Figure 1.



*Figure 1 - A Low Pressure System*

The wind does not, of course, flow directly into or out of the low pressure system. The Coriolis force causes the air to spiral into the system at low level and spiral out at high level. The wind speed and direction thus change with altitude.

The wind observed at, or near, ground level at the launch site is not a true representation of the low level wind. This wind, sometimes referred to as the “surface wind”, is not a wind per se but reflects the results of the interaction between the low level wind and surface features. Its effects are generally not felt more than a few hundred feet above the ground, but are very significant at ground level.

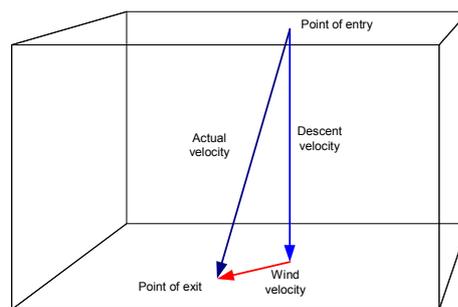
<sup>2</sup> Chapter 6 of “Atmosphere, Weather and Climate” by Barry and Chorley gives a good description of winds and the forces that drive them.

Surface wind is affected by the shape of the terrain. When it encounters rising ground the wind accelerates, an effect predicted by Bernoulli's equation. Close to the ground the wind forms eddies and turbulence due to its interaction with obstructions such as trees, hedges and buildings. These friction effects can cause local and often large variation in wind speed and direction which strongly influence the first few seconds of flight. Weathercocking, and the subsequent launch angle of the rocket, will be dictated by the surface wind.

## The Wind Model

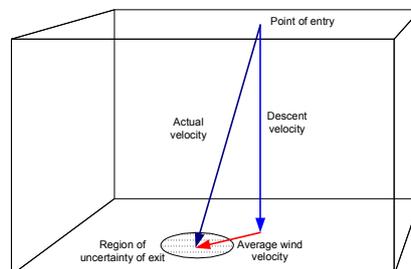
It is mathematically complex to describe the path that a descending rocket will follow from apogee. A simpler approach is to consider its behaviour over short distances and apply some statistics to its behaviour.<sup>3</sup> By summing these effects over the whole descent path we can obtain a reasonable model for the whole descent.

Imagine a rocket falling through a thin slice of the atmosphere. Over that short distance the velocity of the wind and the rocket can be approximately represented as constant values. The rocket will be subjected to a crosswind, so the true path will not be vertical but will follow the vector sum of the descent velocity and wind velocity. If we consider the whole descent from apogee as a sequence of  $n$  layers we can establish the approximate touchdown point.



*Figure 2 - An Ideal Wind Layer*

In practice, the wind vector will not be constant in each layer. We can refine the model by assuming that the magnitude of the vector has a mean value, obtained from a reliable source and a standard deviation. The exit from the layer is thus not a point but a probability distribution based around that point.



*Figure 3 - A Practical Wind Layer*

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<sup>3</sup> SI Adelfang of Marshall Spaceflight Centre has published some interesting papers on the subject of wind profiles and statistics. There are also some very interesting public domain papers on parachute drift from the AIAA and US Dept of Defense.

If we assume that the amplitude of the wind velocity vector is normally distributed about the means speed, and assume a standard deviation, we can link the size of the ellipse to the probability of the exit point lying within that circle. The linkage is:

Diameter	Probability
1 SD	68.2%
2 SD	95.4%
3 SD	99.7%

Figure 4 - Normal Distribution

Why select a normal distribution? Two reasons: firstly many natural occurrences exhibit this distribution, and secondly because of its mathematical convenience.

By considering the effect of descent through  $n$  layers we can establish an ideal touchdown point, and the probability of the actual touchdown point lying within a ellipse around this point.

A decision was made to use 100ft layers in the model. The sensitivity of the model to different layer thicknesses was tested for descents from over 20,000ft. The model was found to be relatively insensitive to change once the thickness was below 500 ft. A thickness of 100ft was chosen as this corresponded to one flight level.

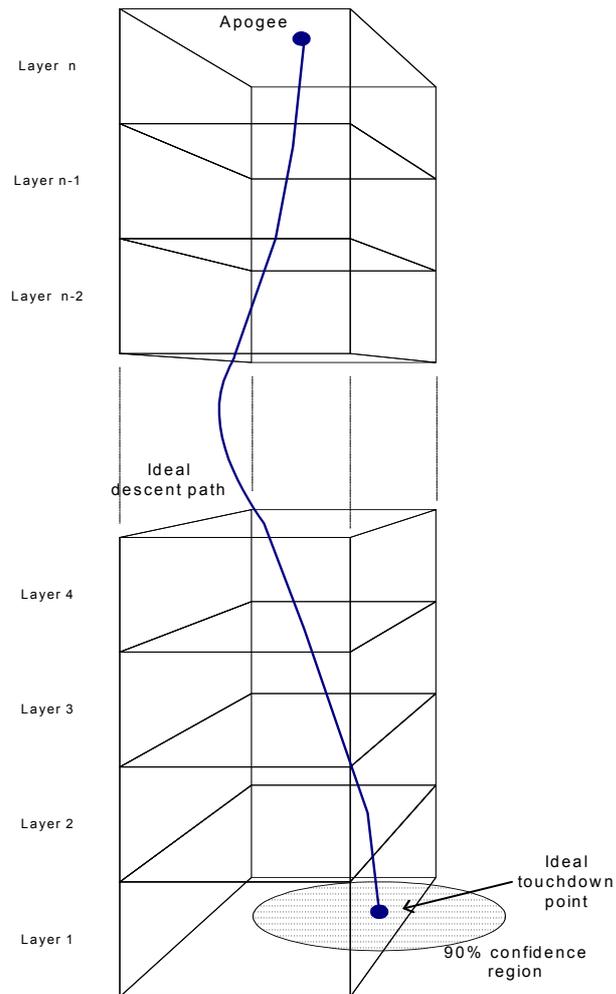


Figure 5 - Drift Through Multiple Wind Layers

## Mathematical Model

This section derives the equations of motion for a rocket descending through a layer. It considers vertical and horizontal motion separately, and gives the general equations that can be used in coding the model.

### Forces

The drag force  $D$  on a body can be calculated from:

$$D = \frac{1}{2} \rho A C_D v^2$$

where  $\rho$  is the density of the air,  $A$  is the surface area presented to the airflow,  $C_D$  is the coefficient of drag and  $v$  is the velocity of the airflow. This equation can be used to find the equilibrium descent velocity where the resistance due to the drag of the parachute(s) and rocket body exactly balance out the weight of the rocket.

### Vertical Motion

A rocket of mass  $m$  kg has a weight of  $mg$  Newtons. As the rocket descends under parachute the weight and drag forces will reach equilibrium such that:

$$mg = \frac{1}{2} \rho A C_D v^2$$

A rocket with CPR can be modelled as several different components, each with its own surface area  $A_k$  and drag coefficient.  $C_{Dk}$ . As these components are linked they will all descend at the same velocity  $v$ , and can be treated as a single mass  $m$ . The equilibrium descent velocity of a rocket comprising  $n$  components can thus be calculated from:

$$mg = \frac{1}{2} \rho v^2 \sum_{k=1}^n A_k C_{Dk}$$

Thus the descent velocity  $v$  can be found from:

$$v = \sqrt{\frac{2mg}{\rho \sum_{k=1}^n A_k C_{Dk}}} \dots\dots\dots(1)$$

If we know, or can reasonably estimate, the mass of the rocket, the surface areas and drag coefficients of the components, and the variation of air density with altitude, we can calculate the equilibrium velocity at any altitude. Appendix 1 shows a representative drag model of a rocket that was used for coding the model. It is reasonably representative of a CPR rocket falling under a drogue parachute. Care should be used when using this model for a rocket falling as two sections connected by a shock cord; rockets in this mode tend to tumble rather than achieve a stable(ish) attitude.

The model assumes that the atmosphere can be sliced into horizontal layers, and that all the atmospheric parameters such as density, wind speed and wind direction are constant within that layer. This allows us to calculate the descent velocity in that layer and, since distance = rate x time, the time it takes to fall through that layer.

In the  $i^{\text{th}}$  layer, where the air density is  $\rho_i$  and the layer is  $d_i$  meters thick, the descent velocity  $v_i$  can be calculated from equation 1:

$$v_i = \sqrt{\frac{2mg}{\rho_i \sum_{k=1}^n A_k C_{Dk}}}$$

Hence the time  $t_i$  it takes to descend through the  $i^{\text{th}}$  layer is:

$$t_i = \frac{d_i}{v_i} = d_i \sqrt{\frac{\rho_i \sum_{k=1}^n A_k C_{Dk}}{2mg}}$$

### Horizontal Motion

The horizontal velocity of the rocket is assumed to be that of the wind in that layer. During time  $t_i$  the rocket is carried by the wind in that layer and will move with that layer. Analysis of the wind speed in each layer thus allows us to determine the direction and distance that the rocket moves when transiting that layer.

The wind velocity vector is not assumed to be constant in each layer but is assumed to vary in both amplitude and direction. The timescales for these variations are assumed to be shorter than the total descent time for the rocket. Mean wind speed and direction at altitudes of 1000, 2000, 5000, 10000, 18000 and 24000 ft can be obtained from the Met Office F214. This data is linearly interpolated to produce mean wind speed and direction for 100ft thick slices in the atmosphere. These mean values are used to produce the normal distribution for speed and direction.

There are three main variables, each of which varies about a mean value with a normal distribution:

- Mean wind speed
- Mean wind direction
- Variability

Mean wind speed and direction are subject to two effects: *randomness* and *variability*. Randomness contributes small changes to both these variables and is associated with the small-scale changes in wind profile, whereas variability contributes larger scale changes and is associated with the turbulence in the boundary region between high and low level winds.

Randomness for a mean wind speed  $v$  is implemented as a standard deviation of  $\sigma = 1.33\text{m/s}$  for up to FL180 and  $\sigma = 2\text{m/s}$  above FL180. These figures are Met office data for 24 hour forecasts<sup>4</sup>. Similarly the standard deviation for a wind direction is  $\sigma = \pm 15$  degrees. Thus for normally distributed variations in wind we would expect that 99.7% of speeds would lie between 0 and  $2v$  m/s and within  $\pm 45$  degrees of the mean direction. There would be a 0.15% probability of the randomness allowing the wind speed to reverse.

Variability is a more complex phenomenon to model. In the software it is treated as a logical variable inasmuch as it is either on or off. If the F214 states direction as “VRB” then variability is “on” and the statistics of variability come into play. Variability is treated in the model as randomness with a much greater standard deviation. The variability for speed has  $\sigma = v$ , thus allowing wind speed to vary between  $-2v$  and  $4v$ . Similarly variability has a direction standard deviation of  $\sigma = \pm 60$  degrees, allowing a variation of  $\sigma = \pm 180$  degrees. Consideration was also given to introducing a vertical component to variability by varying the descent rate, but this was shown to have little overall impact on the scatter plot, possibly because the

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<sup>4</sup> Information on the accuracy of forecasts was obtained from The Met Office.

turbulent region is transited quite quickly as rockets tend to be descending under drogue at that time.

The standard deviations for randomness and variability are two of the key parameters that can be “tuned” in the model. As real data becomes available these parameter can be adjusted to provide a closer match.

### Air Density in the Troposphere<sup>5</sup>

Some models treat air density  $\rho$  as a constant. This is a reasonable assumption for altitudes below 3000 ft as the decrease in density is only a few percent. For high altitude flights the decrease in  $\rho$  can be significant, for example it decreases by 50% by 24000 ft. The variation of the density of the atmosphere with altitude is one of the key factors in calculating aerodynamic forces at different altitudes. It can be shown that the air density at any altitude  $h$ , denoted  $\rho_h$ , varies according to the equation:

$$\rho_h = \rho_0 \left( \frac{T_h}{T_0} \right)^{\left( \frac{g}{LR} - 1 \right)}$$

The variables in this equation are:

$\rho_0$  = Air density at mean sea level (kg/m<sup>3</sup>)

$\rho_h$  = Air density at an altitude of  $h$  meters (kg/m<sup>3</sup>)

$T_0$  = Air temperature at sea level (K)

$T_h$  = Air temperature at an altitude of  $h$  meters (K)

$h$  = altitude AMSL (m)

$g$  = acceleration due to gravity = 9.81 (m/s<sup>2</sup>)

$L$  = lapse rate of temperature with altitude (K/m)

$R$  = characteristic gas constant (J/kg-K)

Note that all altitudes and temperatures are above mean sea level (AMSL) not ground level. This is significant as rocket altitudes are referenced to ground level. The programme offsets the density calculations to launch site altitude. The value of  $T_h$  can be calculated from  $T_0$  and the lapse rate  $L$ :

$$T_h = T_0 - hL$$

Which leads to a usable equation linking air density to altitude:

$$\rho_h = \rho_0 \left( \frac{T_0 - hL}{T_0} \right)^{\left( \frac{g}{LR} - 1 \right)}$$

Some of these constants may vary with location, so the air density spreadsheet allows users to change the data. Typical values are:

$\rho_0 = 1.225 \text{ kg/m}^3$  (may vary with location)

$T_0 = 288.15 \text{ K}$  (may vary with location)

$g = 9.81 \text{ m/s}^2$

$R = 287.05 \text{ J/kg-K}$

$L = 0.008 \text{ K/m}$

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<sup>5</sup> It was a conscious decision to avoid the “standard atmosphere” model and allow this model to be tuned.

Lapse rate,  $L$ , differs between dry air and moist air. It is higher in dry air and lower in saturated, typical values are 0.01 K/m (dry) and 0.005 K/m (saturated). Moist air generally produces cloudy weather, and rockets are not launched into cloud, so a typical value of 0.008 K/m is suggested for the model.

The value of  $T_0$  can be measured at the launch site. If measured in centigrade ( $^{\circ}\text{C}$ ) the value can be converted to Kelvin (K) by adding 273.15.

## The Software

This section describes the software. It starts with a brief functional description of the software, then describes the inputs and outputs from the current stable version 1.5.

### Flowchart

The basic functionality can be described from a flowchart.

The program commences by dimensioning variables and arrays, reading in data from the spreadsheet using the “cells(x,y)” command, and initialising variables. It uses this information to calculate the mean wind profile from mean sea level to FL240. This information is used to create the spreadsheet and graphs “plot 1 – wind profile”.

The next step is to do a first pass through the program without adding any randomness or variability. This produces a mean descent path which is used to create the spreadsheet and graphs “plot 2 – descent”. Note that apogee, main deployment and launch site (landing) altitude are referenced to AGL but wind profile is referenced to mean sea level. The program takes account of this difference.

The remaining passes add randomness and variability and are used to create the scatter plot on the “input & output” spreadsheet.

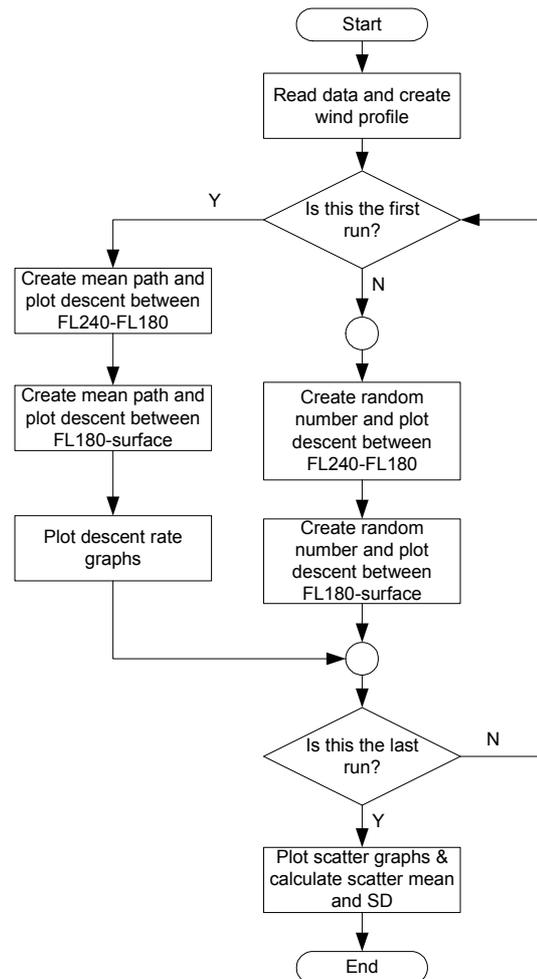


Figure 6 - Flowchart

### Inputs

The program gets its inputs by reading from the spreadsheet. The wind profile data is typed into the “F214 Input” area of the spreadsheet. The data comes from the Met Office aviation forecast, a free service which requires registration. From the met office homepage [www.met-office.gov.uk](http://www.met-office.gov.uk) click on “Aviation”, log in, press “Launch” and open the F214 for the period you require.

Note that the F214 covers a 6 hour period, and is normally available for 12-18 hours ahead.

With the F214 open, locate the grid point nearest your launch site and you should see a table similar to Figure 7. The first columnist ha altitude in thousands of feet, the second gives the heading, the third gives the wind speed and the fourth give the air temperature in degrees Centigrade. Copy the heading and speed columns into the spreadsheet. If the wind heading is shown as “VRB”, which means variable, type in the value of either adjacent layer (it’s not too critical which you use) and set the corresponding variable column to “1”. This triggers variability within the program.

170 20 +08	02 130 11
01 120 11	
EIDW * 5W *	
5230N 05W	
14 24 050 25 -34	523
18 18 050 20 -19	24 0
13 10 030 15 -03	18 0
15 05 090 20 +02	10 0
14 02 VRB 05 +03	05 1
15 01 110 10 +04	02 0
	01 0
50N 05W	
14 030 35 -34	50N
8 040 35 -19	24 04

F214 Input			
Alt (k ft)	Heading	Variable	Speed
24	50	0	25
18	50	0	20
10	30	0	15
5	90	0	20
2	110	1	5
1	120	0	10
0	120	0	4

Figure 7 - F214 Transferred to Spreadsheet

In the “Descent Input” area of the spreadsheet you should put the apogee altitude. Apogee is measured above ground level (AGL) so it can be referenced to the launch site. For normal motor ejection set the main chute altitude to apogee, or whatever altitude you expect the main chute to deploy. If the rocket uses CPR you should also set the main chute altitude to match the altimeter settings. A CPR entry can be seen in Figure 8.

Descent Input		
Apogee altitude	24000	ft AGL
Main chute altitude	1000	ft AGL

Figure 8 - Rocket Descent Inputs

The “Rocket Data” area is used to calculate the weight and drag forces acting on the rocket during descent. The rocket dry mass is used to calculate the weight of the rocket. Rocket diameter, length and fin area are used to estimate the total drag from the airframe after deployment, the method of estimation is described at Appendix 1.

The drogue and main chute diameters and drag coefficients are used to estimate the drag added by each chute as it deploys. If the rocket uses motor ejection, or uses CPR but without a drogue, you should set the drogue diameter to zero.

Rocket Data		
Rocket dry mass	8	Kg
Rocket diameter	3	inches
Rocket length	70	inches
Area of one fin	18	sq inches
drogue chute diameter	0	inches
Drogue CD	1.5	
Main chute diameter	60	inches
Main chute CD	2	

Figure 9 - Rocket Data for Drag Model

The “Prog Data” area contains two variables. The first is the number of points to be plotted which should be an integer between 2 and 200. Setting this to a larger number requires some adjustment to both the program and the spreadsheet otherwise misleading results and plots will be produced.

The second data input is the launch site altitude in feet AMSL. This is used to calculate the apogee and landing points AMSL.

<b>Prog Data</b>		
Number of points	200	(max 200)
Launch site altitude	750	ft ASL

*Figure 10 - Other Program Data*

Note that the wind model is only valid to 24,000 ft AMSL, so the total of the launch site altitude and apogee altitude must not exceed 24,000 ft. A warning is displayed in the spreadsheet if this condition occurs.

A final set of inputs can be found in the “Atmospheric Density Model”. These should normally be left untouched, but may be adjusted if launch weather conditions are abnormal or you plan to launch on another planet.

Sea level pressure	1013.25	mb	<p style="text-align: center;"><u>NOTE</u></p> <p>The values of the atmospheric parameters can be changed in this spreadsheet, and they will automatically be picked up and read by the model.</p>
Sea level density	1.25	kg/m <sup>3</sup>	
Sea level temp	288.15	K	
Lapse rate	8	K/km	
		J kg <sup>-1</sup>	
Gas constant (dry air)	287.04	K-1	
Acceleration due to gravity	9.80665	m s <sup>-2</sup>	

*Figure 11 - Atmospheric Density Model*

Once you’re satisfied with the inputs, press the “Press to Run” button. The data in the “Results” area should change and a new scatter graph produced in about 2-3 seconds.

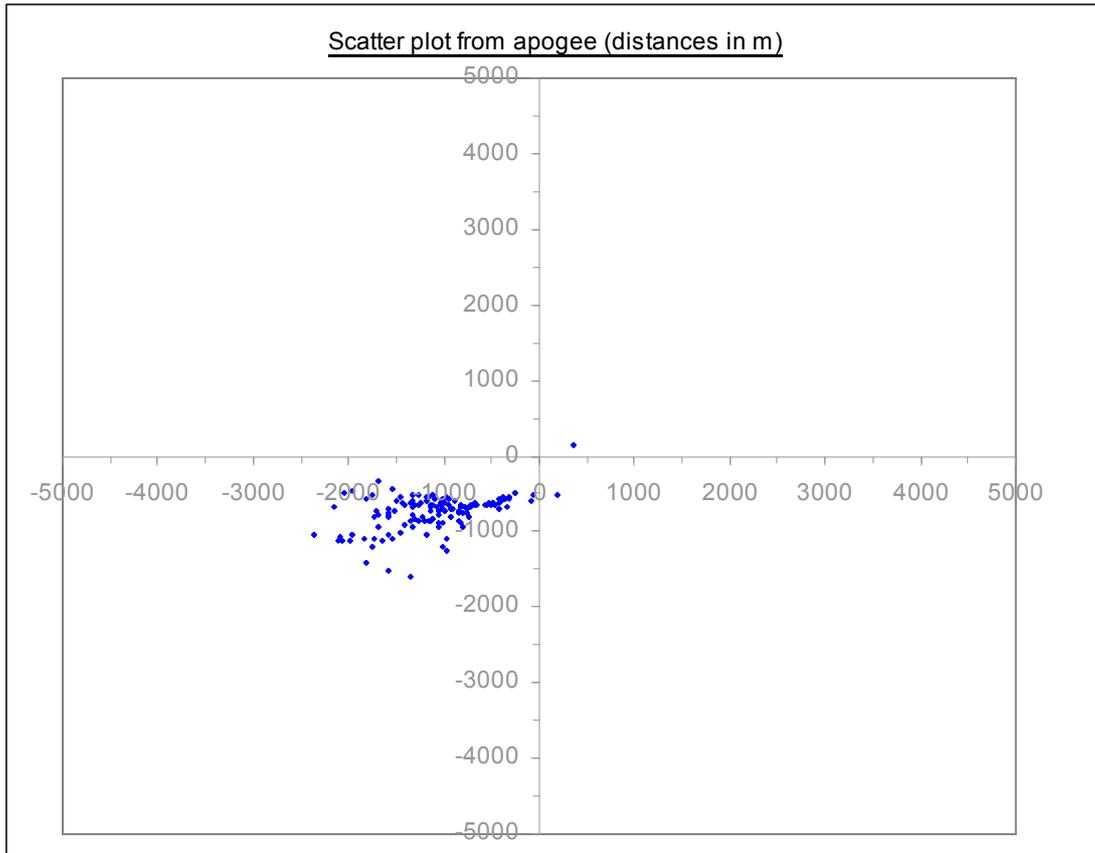
### Outputs

The outputs are written to the spreadsheets using the “cells” command in VBasic. They are produced as graphs, although you can take the data from the cells for further manipulation or analysis.

The main plot is the scatter plot. This shows the distribution of landing points around the sub-apogee point NOT THE LAUNCH POINT. This data is analysed in the “Results ±3σ” area to show the statistical mean, min and max values of range and bearing.

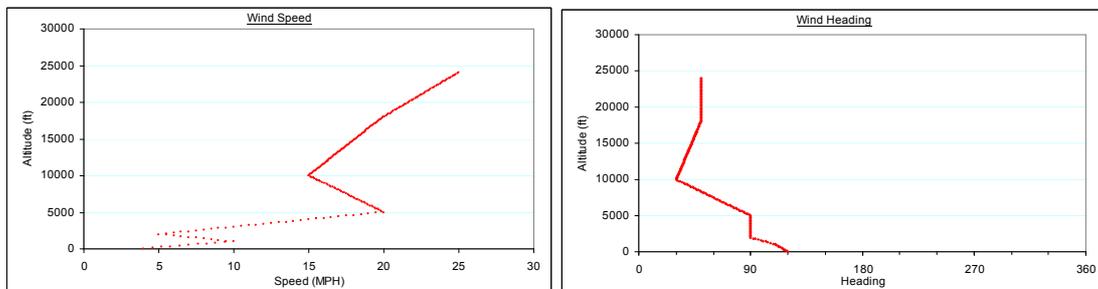
A scatter plot can be seen in Figure 12.

The scatter plot has been set to a scale of ±5km N-S and ±5km E-W. The scale can be adjusted by right clicking on the X and Y axes in turn, selecting “format axis” and setting the max and min values under the “scale” tab.



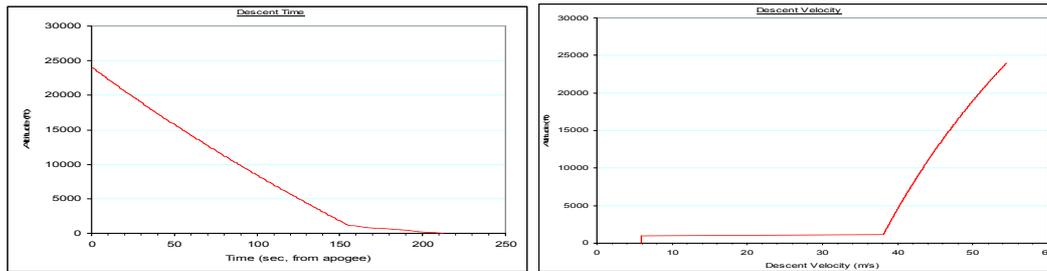
*Figure 12 - Scatter Plot*

The plots of wind profile can be seen in the "Plot 1- Wind Profile" sheet. Typical plots can be seen in Figure 13.



*Figure 13 - Wind Profile Plots*

A further pair of graphs can be seen in the "Plot 2- Descent" sheet. These show the descent time and descent velocity in seconds and m/s respectively. They are provided mainly because the data was available, but may help to give an idea of how long the rocket should be airborne after apogee. Typical graphs for a CPR flight can be seen at Figure 14.



*Figure 14 - Descent Plots*

## Next Developments

Having created the model the next step is to calibrate it. This comprises taking data on real HPR flights to significant altitudes and using these to tune the variables. The general approach will be to take F214 and launch data from real rocket launches and adjust the variables to fit the data.

There are several variables that could be adjusted, most notably the standard deviations for wind speed and direction above and below FL180 (4 variables) and the scaling factors in the random number subroutines (2 variable). The standard deviations were set based on information from the Met Office and should not require much adjustment. The focus of the tuning the model should thus be on adjusting the two scaling factors for the random number subroutines.

This parametric tuning may also account for some of the assumptions, in particular the assumptions that a rocket moves with the air in every layer and descends at a steady rate.

The model only considers descent from apogee, and does not consider any aspects of the launch or ascent. Further developments may extend the scope of the model to cover the whole flight. These extensions include:

- Integration of the scatter plot with map data to aid visualisation
- Flight analysis in 5 DOF (assuming the effect of rotation around the roll axis is insignificant) based on a given weathercock angle.
- A detailed analysis of weathercocking in 6DOF, taking into account the changing mass distribution of the rocket.
- Consideration of failure modes, their probabilities, and their impact on landing site.

The end result would be a program that could predict the scatter plot for all conditions, including rocket failures. This would be a powerful tool for assisting RSOs to make go/no-go decisions on high altitude flights. It would also be useful for clubs to determine the best launch point, and perhaps set altitude limits depending on prevailing conditions. Ultimately, it will help to maintain the good safety record of rocketry in the UK while we continue to push the limits.

## APPENDIX 1 – A SIMPLE DRAG MODEL OF A DESCENDING ROCKET

If we ignore the drag from the shock cord, we can produce a simple model of a CPR rocket. Such a rocket has the following components:

- Main chute
- Drogue chute
- Main body
- Fins

The layout of a typical rocket descending is shown in the diagram below. This

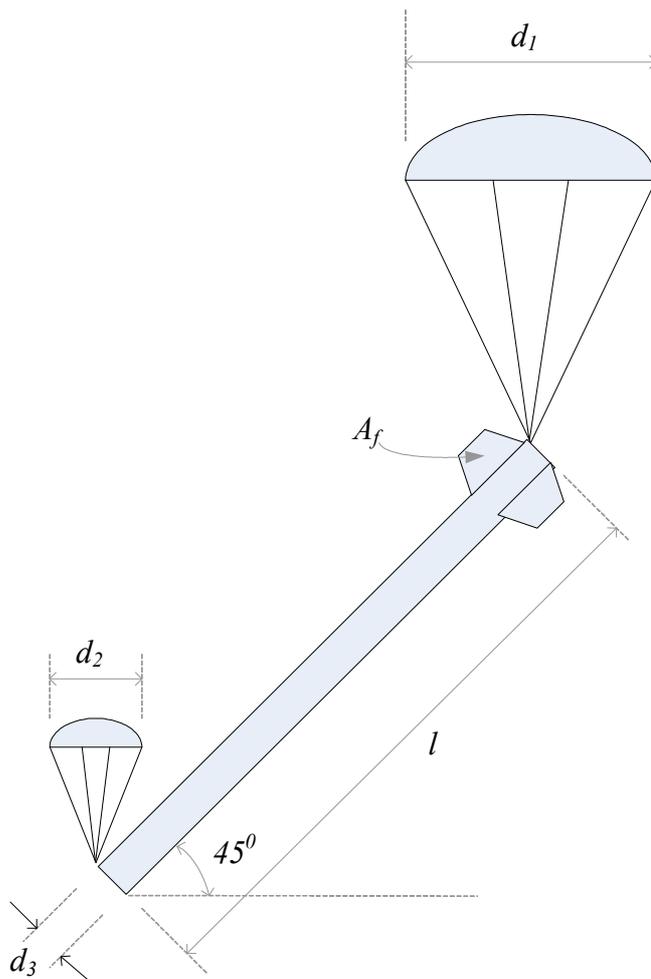


diagram serves to explain how the various components of the rocket add to the total drag.

The main and drogue chutes have the area of circles of diameters  $d_1$  and  $d_2$  respectively. This ignores spill holes. Construction techniques may result in different drag coefficients so these can be input via the spreadsheet. It is possible to “free fall” with no drogue by making the diameter  $d_2=0$ .

For simplicity, we assume that the main body of the rocket approximates to a cylinder of length  $l$  and diameter  $d_3$ . The surface area of this cylinder (ignoring ends) is:

$$A_3 = \pi \frac{d_3^2}{4}$$

Of course, the whole surface is not exposed to the wind.

Simple geometry shows that

only  $1/\pi$  of the area contributes to drag. As the main chute provides more drag than the drogue then the main body will be inclined to the horizontal. This inclination is assumed to be 45 degrees, thus the area presented to the airflow is only  $0.7071/\pi$  of the total area.

The fin area represents the area and number of fins that are exposed to the airflow. This is multiplied by 0.7071 as the fins are inclined at 45 degrees to the airflow.

A table summarizing the area contributions of each part of the rocket that are used for calculating drag can be seen below.

k	Component	Assumed area $A_k$	Assumed $C_{Dk}$	Remarks
1	Main chute	$A_1 = \frac{\pi d_1^2}{4}$	1 Note 1	Diameter $d_1$ m
2	Drogue chute	$A_2 = \frac{\pi d_2^2}{4}$	1 Note 1	Diameter $d_2$ m
3	Main body	$A_3 = 0.7071 \times l \times \frac{d_3^2}{4}$ Note 2	1	Treat as a cylinder diameter $d_3$ and length $l_3$ .
4	Fins	$A_4 = 0.7071 \times 1.66 A_f$ Notes 2,3	1	There are 3 or 4 fins and the area of each fin is $A_f$ .

*Note 1.* The values of CD for the main and drogue parachutes can be varied as inputs in the spreadsheet.

*Note 2.* Only  $1/\pi$  of the surface is exposed to the airflow. It is also assumed that the body does not hang vertically but at 45 degrees, thus only  $\cos(45) = 0.7071$  of the exposed area contributes to drag.

*Note 3.* The coefficient 1.66 is derived from the aspect ratio of the fins. For 4 fins the area presented lies between  $2 A_f$  and  $2 \times \cos 45 = 1.414 A_f$ . For 3 fins this aspect lies between  $1 + \cos 60 = 1.5 A_f$  and  $2 \cos(30) A_f = 1.73 A_f$ . The average of these is  $1.66 A_f$ .

## APPENDIX 2 – GENERATING NORMALLY DISTRIBUTED RANDOM NUMBERS

VBasic has the ability to generate random numbers between 0 and 1 with a flat distribution. A technique was needed to generate normally distributed random numbers with a mean of 0 and standard deviation of around 0.3.

Two techniques were considered:

- Generate the random numbers in Excel and read them into VBasic
- Write a routine in VBasic to generate the random numbers.

Normally distributed random numbers can be generated in any cell in an Excel spreadsheet using the function NORMDIST. The form of the command is:

*= Normdist (Rand(), mean, variance)*

Since the variance is the square of the standard deviation it is easy to produce random numbers on demand. The cell is refreshed whenever anything is written to the spreadsheet, either from the keyboard or from VBasic. The VBasic function thus needs two lines: the first to write to any cell to trigger a new random number and the second to read the new random number into a VBasic variable.

The advantages of this approach are simplicity and the ability to easily tune the mean and standard deviation of the random number. The big disadvantage is that it is very slow.

An alternative approach was adopted. This exploits a useful property of statistics called the central limit theorem. This theorem states that *the distribution of an average tends to be Normal, even when the distribution from which the average is computed is non-Normal*. In other words, if we take the mean of a number of random numbers then the mean will lie on a Normal distribution.

A short subroutine was written to generate 30 flat-distributed random numbers in VBasic and find their mean. Tests showed that the distribution of numbers generated this way was normal, with a mean of 0.5 a variance of 0.083 and a standard deviation of 0.288.

This distribution was centred on zero by subtracting 0.5 from all the generated random numbers. This value was then returned as a normally distributed random number. This subroutine reduced the run time of the programme by a factor of 12, and was thus adopted.