

Forces on a Rocket

In order to understand the behaviour of rockets it is necessary to have a basic grounding in physics, in particular some of the principles of statics and dynamics. This section considers the forces acting on a rocket, and how they affect its performance. In particular, the section looks at the linear and rotational behaviour of a rocket.

Forces

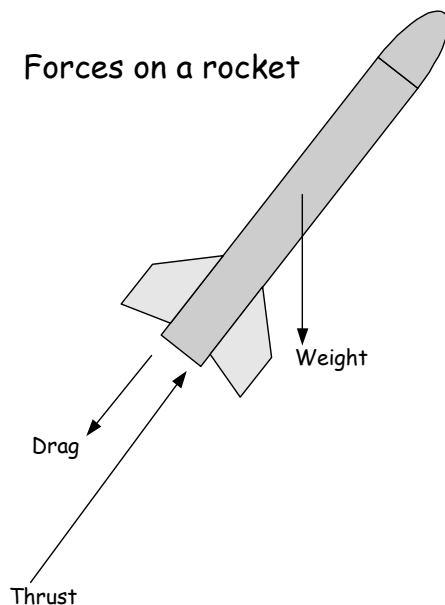
Newton's second law leads to the well known equation:

Force = mass x acceleration

Expressed mathematically it can take one of several equivalent forms:

$$F = ma = m \frac{dv}{dt} = m \frac{d^2r}{dt^2}$$

Most basic text books leave the concept of force at this point, possibly illustrating the idea of acceleration as being a change of velocity over time. The conventional view of a model rocket is to consider three forces acting on a rocket: thrust, drag and weight.



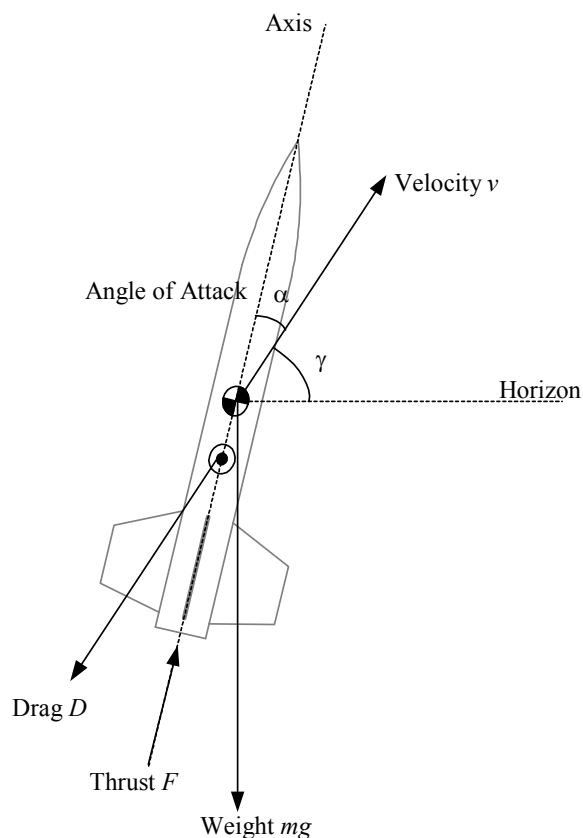
The forces on a rocket vary throughout the flight. At a basic level the weight reduces as propellant is consumed, the thrust changes depending on the burn profile, and drag increases with the square of the velocity. Most model rocket flights take place within a few thousand meters of the Earth, however in higher altitude flights other “constants”

start to change. Air density, used to calculate drag, changes with temperature and hence with altitude. Even gravity reduces slightly as a rocket moves away from the Earth. Our Newton's second law equation starts to look complex as it contains nothing which is constant.

Motion in the direction of flight

The motion of a rocket can be predicted if we can understand how the acceleration varies with time. Knowledge of the rocket's acceleration allows us to calculate its velocity and altitude through calculus. If we can arrive at a generalised equation for how acceleration varies with all the other possible factors, we can apply this to any rocket, motor, flight profile, atmosphere or even planet and predict how our rocket will behave.

Let's consider a rocket in the atmosphere flying at some angle to the horizon and with an angle of attack to the airflow. We will regard the rocket as having a fixed motor so that thrust is always aligned with the axis of the rocket. The forces acting on the rocket, their directions and apparent locations are shown on the diagram below.



Note that the thrust acts along the axis, aerodynamic forces act through the centre of pressure, and weight acts through the centre of gravity. As acceleration acts through the

centre of gravity, the diagram shows velocity acting through that point. If we consider all the forces, and components of forces, acting in the direction of the instantaneous velocity vector (direction of flight):

$$m \frac{dv}{dt} = F \cos(\alpha) + mg \cos(90 + \gamma) - D$$

Rearranging this equation, and assuming that the angle of attack is small (less than 10degrees) we arrive at an equation for the acceleration in the direction of flight.

$$\frac{dv}{dt} = \frac{F}{m} - g \sin(\gamma) - \frac{D}{m} \quad \text{..... equation 1}$$

Let's consider this equation. The thrust force acts in a forward (positive) direction and the gravity and drag terms act to oppose it (negative direction). The acceleration is positive if the F/m component is greater than the other two, which happens when the thrust is larger than the combination of weight & drag. If the thrust term is zero, then the acceleration is negative, which makes sense as the rocket will be slowing down. In horizontal flight the weight component is at right angles to the direction of travel, so it introduces no acceleration in that direction.

This gives us an equation for the acceleration in general terms, but many of the terms are also variable. Mass m decreases as fuel is burned. The thrust F changes throughout the burn. Drag D changes with the square of the velocity v . To make use of this general equation we need to know how each of the terms varies with time. In practice these equations are extraordinarily complex to solve, and the general approach is to solve them using numerical methods such as linear extrapolation and Runge-Kutta's method.

Linear extrapolation treats the equations of motion as being linear over very short time intervals. The values of distance, velocity, acceleration and all the forces are initially zero. Over a very short time interval the acceleration, forces, air density and other factors are assumed to be constant. This allows calculation of the values at the end of this time interval, and establishes the values for the start of the next time interval. By repeating this over many time intervals it is possible to approximate the values at any time in the future. It will be obvious that this process results in accumulation of errors over a period of time as small errors will be carried forward into all future calculations. In order to increase the accuracy of results over long periods the time intervals must get smaller. Eventually the method collapses under the weight of computational intensity. It is, however, quite simple to understand and lends itself to spreadsheets.

Runge-Kutta's method is a numerical technique for solving first and second order differential equations. It is less computationally intensive than linear extrapolation, and is generally more accurate, but requires a firmer grasp of mathematics. It is not normally taught until second year on engineering degree courses.

Motion perpendicular to the direction of flight (Pitch)

Having considered the motion along the direction of flight, we'll now take a look at the motion at right angles to the direction of flight. The forces in this direction are in equilibrium if the rocket is flying straight, thus accelerations are zero. The first gust of crosswind soon knocks the rocket out of equilibrium and we then have to consider the restorative forces from the fins, forces from the wind, and any components of thrust which are no longer along the direction of flight.

The rocket will tend to rotate about its centre of gravity, with all the aerodynamic forces acting through its centre of pressure. Forces which do not pass through the centre of gravity, which is by definition the axis of rotation, are called torques. A torque is the force multiplied by the distance at which it acts from the axis of rotation.

The forces are thus "torques" about the centre of gravity, and since they are trying to alter the direction of flight (pitch) we call them moments of pitch. Physics purists will criticise this simplistic explanation, however it will suffice for our purpose.

Newton's second law allows us to analyse linear motion but needs to be adapted to analyse rotational motion. As shown earlier, the basic equation that we use for linear motion is:

$$F = m \frac{d^2 r}{dt^2}$$

Torques cause a rotational acceleration which is measure in degrees per second per second or more usually in radians per second per second, where 360 degrees is 2π radians. In linear motion we have mass, which has an inherent inertia which resists acceleration. It requires more force to accelerate a heavy object than a light one. In rotational motion the equivalent of mass is called the moment of inertia, denoted I .

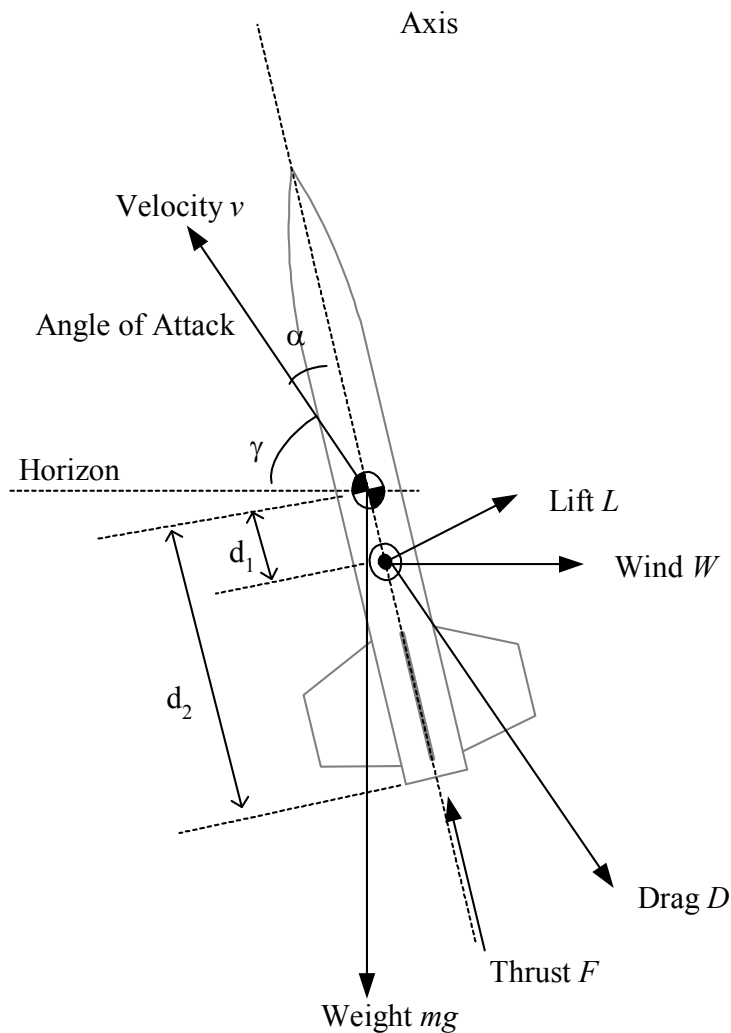
There is an equivalent form of this equation when we consider rotational motion. We say that:

Torque = moment of inertia x angular acceleration

Or mathematically:

$$\tau = I \frac{d^2 \theta}{dt^2}$$

The equivalence of the linear and rotational equations should be obvious. In place of force F we have torque τ , in place of mass m we have moment of inertia I , and in place of distance r we have angle θ . Moment of inertia is a relatively straightforward idea. We define moment of inertia as the sum of all the masses which comprise a body multiplied by the square of the distances:



From the diagram it can be seen that the forces causing an angular acceleration about the centre of gravity are aerodynamic: the lift due to the fins and any torque inducing forces such as crosswinds. These act through the centre of pressure. These torques combine to overcome the moment of inertia and cause an angular acceleration of the rocket.

It is useful to express this mathematically in a generalised form, that is an equation involving all terms. This allows various forms of the equation to be derived, for example versions with and without thrust or wind, positive or negative or zero angles of attack etc. For convenience, we'll define a clockwise rotation as being positive and the direction of the velocity vector as the reference direction for all angles:

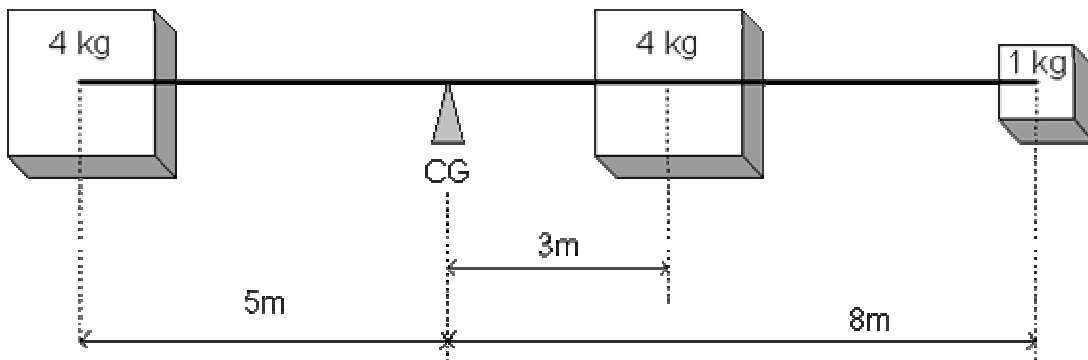
$$I \frac{d^2 \alpha}{dt^2} = Fd_2 \sin(\alpha) - Ld_1 - Wd_2 \cos(\gamma) \dots\dots \text{equation 2}$$

This is an important equation, and we'll have a look at each of the terms in this equation before deriving a more informative version.

Moment of Inertia. The moment of inertia of a body is a function of its mass distribution. In some texts it is referred to as the “rotational inertia” or as the “second moment of mass distribution”. We define moment of inertia as:

$$I = \sum_{i=1}^k m_i r_i^2 \dots\dots \text{equation 3}$$

If we have a body which comprises k pieces, and each of these has a mass of m_i kg and is located r_i metres from the bodies centre of gravity, we can calculate I by adding up the total of all the $m_i r_i^2$ for the whole body. As an example, if a body comprises 3 blocks, two of which have masses of 4 kg and one has a mass of 3 kg. They are linked by a lightweight beam as shown below:



The moment of inertia is:

$$I = \sum_{i=1}^3 m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = 4 \times 5^2 + 4 \times 3^2 + 1 \times 8^2 = 200 \text{ kg m}^2$$

It can be seen from the example what is meant by mass distribution. If we move the masses around we don't change the overall mass of the system, but we will change their location and hence the moment of inertia. To calculate the moment of inertia of a rocket we need to know the mass, centre of gravity location, and distance from the rocket's centre of gravity of all its constituent pieces. Armed with this knowledge we can easily calculate I .

Lift. The section on aerodynamics shows that the lift of the fins can be calculated from:

$$L = \frac{1}{2} \rho A_F C_L v^2$$

For small angles of attack (less than 10 degrees) the coefficient of lift C_L of a flat fin is proportional to the angle of attack. We can thus define a constant k_L for small angles of attack such that:

$$C_L = k_L \alpha$$

Thus

$$L = \frac{1}{2} \rho A_F k_L v^2 \alpha \dots\dots\dots \text{equation 4}$$

Reconciliation

If we substitute equation 4 into equation 2 we get:

$$I \frac{d^2 \alpha}{dt^2} = \frac{1}{2} \rho A_F k_L v^2 d_1 \alpha - W d_2 \cos(90 - \gamma) + F d_2 \sin(\alpha)$$

A useful device called the “small angles approximation” tells us that for small angles of attack $\sin(\alpha) \sim \alpha$, as long as the angle is measured in radians. We thus get:

$$I \frac{d^2 \alpha}{dt^2} = \frac{1}{2} \rho A_F k_L v^2 d_1 \alpha - W d_2 \cos(90 - \gamma) + F d_2 \alpha$$

Or with minor rearrangement to collect terms containing α :

$$I \frac{d^2 \alpha}{dt^2} = \left(\frac{1}{2} \rho A_F k_L v^2 d_1 + F d_2 \right) \alpha - W d_2 \cos(90 - \gamma) \dots\dots\dots \text{equation 5}$$

Maths tyros will recognise this as a second order differential equation in terms of α . So what, I hear you ask. Well the solution for a second order differential equation of this form is a sinusoid, as long as the force is acting to restore the rocket to straight flight. This means that if the rocket has an angle of attack the restoring force will cause it to swing into positive and negative angles of attack. The frequency of this oscillation will depend of the moment of inertia I of the rocket, and its amplitude will depend on the lift generated by the fins, whether thrust F is applied and the distance d_1 between the CP and CG. This bit of maths has just explained why we sometimes see oscillating smoke trails in marginally stable rockets and linked it to the factors which affect the frequency and amplitude of the oscillation.

We've used the small angles approximation quite a bit here, but have never justified it. If you've read any aerodynamics you'll recall that the lift coefficient increases with increasing angles of attack. When it increases beyond a critical angle, typically 12 degrees, the lift coefficient drops sharply as the airflow stalls. At this point the fins cease to apply any aerodynamic force and the rocket becomes unstable. Not only is 10 degrees a convenient mathematical upper limit beyond which $\sin(\alpha) \neq \alpha$, but it also conveniently matches the aerodynamic limits of the fins.