Momentum and the Rocket Equation

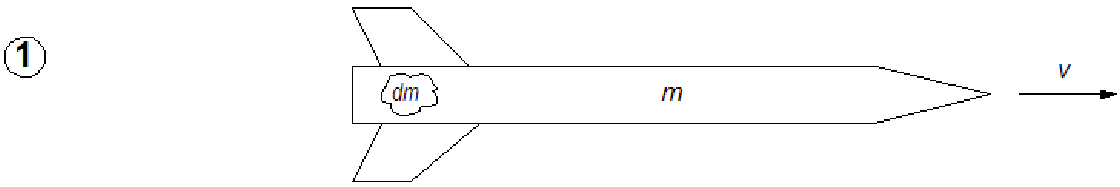
The rocket equation gives an explanation of how the gas ejected from the nozzle is used to propel the rocket forwards. In its classical form it has little practical use in model rocketry, as it assumes no drag and no gravity. It does, however, provide a useful means of understanding the relationships between the mass of the rocket, mass of fuel, exhaust velocity and the "burn out" velocity of a rocket.

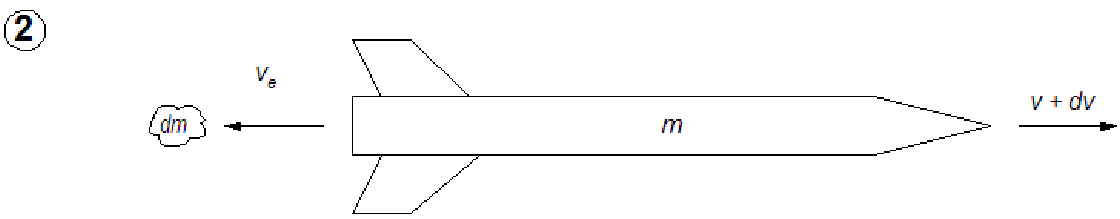
This short paper considers the idea of momentum, and examines how an understanding of momentum can be used to derive the rocket equation. It then examines some of the design "trade offs" in building a rocket.

Momentum

The behaviour of a rocket motor can best be explained by understanding the principle of conservation of momentum.

We can consider a rocket at some point during its flight as having a mass of m with a little bit of fuel of mass dm about to leave the rocket. At this point the rocket has a velocity v. An instant later, the element of propellant has left the nozzle at the exhaust velocity (relative to the rocket) of $-v_e$. Why the minus sign? Well the bit of propellant is travelling in the opposite direction to the rocket, and positive velocities are in the direction of travel of the rocket so anything travelling in the opposite direction is negative.





As a result of ejecting this small bit of propellant the rocket increases its velocity by a small amount, +dv. This is positive because it's in the direction of travel of the rocket. Conservation of momentum tells us that the momentum before the propellant is ejected is the same as the momentum after the propellant is ejected. Thus:

Initial momentum = $(m + dm)v$

Final momentum = $m(v + dv) - dm\, v_e$

As conservation is conserved, the initial momentum and the final momentum are equal, thus:

$(m + dm)v = m(v + dv) - dm\, v_e$

If we rearrange this, ignore second order terms, and integrate over the duration of the burn we find that:

$$v_{final} = v_e \ln\left(\frac{m_{initial}}{m_{final}}\right)$$

Where ln denotes the natural logarithm. This is the classic form of the rocket equation. The initial mass of the rocket is the total structural, payload and fuel mass, whereas the final mass assumes that all the fuel has been burned.

The Rocket Equation

How can we use this equation in practice? Imagine we have a rocket which has a "dry mass" m_r , a payload mass m_p and starts with propellant of mass m_f If the rocket burns the propellant and ejects it from the nozzle at a rate v_e, then the "burn out" velocity v_b obtained by burning all the propellant is:

$$v_b = v_e \ln\left(\frac{m_r + m_p + m_f}{m_r + m_p}\right) \quad \ldots\ldots\text{ equation 1}$$

So what does this tell us? Well for a start, if we have a way of calculating the final, or "burn out" velocity of a rocket which depends on the exhaust velocity, the mass of fuel and the mass of the rocket. The exhaust velocity depends on the design of the motor, and the amount of fuel and mass of the rocket depend on the design of the rocket.

Lets explore this equation further. If the amount of fuel carried is very small compared to the mass of the rocket, then the bit in brackets has a value very close to 1. Since the natural logarithm of 1 is zero (try it on your calculator), we can see that the change in velocity is zero. This makes sense, as a rocket with very little fuel is not going to go very far. Let's consider the other "limiting case", a rocket which is almost entirely fuel. In this case m_f is much greater than m_r so the bit within the brackets gets close to infinite. As the natural log of infinite is a very large number, we end up with a velocity many times the exhaust velocity. This also makes sense, as a long burn will continuously accelerate the rocket to a very high velocity.

It can be seen that there is a relationship between the performance of a rocket and the relative masses of the structure and fuel. To explore this further, we need to

We can define two useful terms which can be used to quantify this relationship: the *payload ratio* and the *structural ratio*. The payload ratio, denoted π, is simply the mass of any payload carried by the rocket to the total mass of the rocket:

$$\pi = \frac{m_p}{m_r + m_p + m_f}$$

The structural ratio, denoted ε, is simply the ration of the structural mass of the rocket to the total mass of the rocket:

$$\varepsilon = \frac{m_r}{m_r + m_p + m_f}$$

Manipulating equation 1 we get:

$$v_b = -v_e \ln(\varepsilon + (1-\varepsilon)\pi)$$

There is thus a maximum velocity that can be obtained by a rocket, and that occurs when there is no payload, in other words when π = 0. This result is intuitively correct. The value of this maximum velocity increases as the structural ratio decreases. This, too, is intuitively correct as the structural ratio with no payload present indicates how much of the rocket is structural. As the remainder of the mass comprises propellant it is easy to see why a low structural ratio results in a higher burnout velocity. A typical satellite launcher comprises about 80% propellant at launch.

MULTISTAGE ROCKETS

So far we've considered single stage rockets, how can we apply this to multistage rockets? The trick is to recognise that all the stages above the one which is burning are its "payload". Each stage thus has a separate payload ratio, and it can be shown that the payload ratio for the whole rocket is simply the product of all the payload ratios. For example, if we have a rocket with 3 stages the payload ratio for the whole rocket is:

$$\pi_{total} = \pi_1 \times \pi_2 \times \pi_3$$

Similarly for a rocket with *N* stages, the payload ratio is:

$$\pi_{total} = \pi_1 \times \pi_2 \times \pi_3 \times \ldots \times \pi_N$$

It is not intended to prove it in this essay, but the burn out velocity of a multistage rocket can be maximised by making the payload ratio of all the stages identical. Practically, such a rocket cannot be readily built as the designer will reach a state for low values of *N* where the stages become so massive that the rocket ceases to be affordable. It can be shown that, for any given exhaust velocity $\varpi\varepsilon$, and payload ratio π, the advantages of more than 4 stages are negligible.